Probability

INTRODUCTION

Random Experiment:

It is an experiment which if conducted repeatedly under homogeneous condition does not give the same result.

The total number of possible outcomes of an experiment in any trial is known as the exhaustive number of events.

For example:

(i) In throwing a die, the exhaustive number of cases is 6 since any one of the six faces marked with 1, 2, 3, 4, 5, 6 may come uppermost.

(ii) In tossing a coin, the exhaustive number of cases is 2, since either head or tail may turn over.

(iii) If a pair of dice is thrown, then the exhaustive number of cases is $6 \times 6 = 36$

(iv) In drawing four cards from a well-shuffled pack of cards, the exhaustive number of cases is ${}^{52}C_4$.

Events are said to be mutually exclusive if no two or more of them can occur simultaneously in the same trial.

For example:

(i) In tossing of a coin the events head (H) and tail (T) are mutually exclusive.

(ii) In throwing of a die all the six faces are mutually exclusive.

(iii) In throwing of two dice, the events of the face marked 5 appearing on one die and face 5 (or other) appearing on the other are not mutually exclusive.

Outcomes of a trial are equally likely if there is no reason for an event to occur in preference to any other event or if the chances of their happening are equal.

For example:

(i) In throwing of an unbiased die, all the six faces are equally likely to occur.

(ii) In drawing a card from a well-shuffled pack of 52 cards, there are 52 equally likely possible outcomes.

The **favourable cases** to an event are the outcomes, which entail the happening of an event.

For example:

(i) In the tossing of a die, the number of cases which are favourable to the "appearance of a multiple of 3" is 2, viz, 3 and 6.

(ii) In drawing two cards from a pack, the number of cases favourable to "drawing 2 aces" is ${}^4C_2$.

(iii) In throwing of two dice, the number of cases favourable to "getting 8 as the sum" is 5, : (2, 6), (6, 2), (4, 4), (3, 5) (5, 3).

Events are said to be independent if the happening (or non-happening) of one event is not affected by the happening or non-happening of others.

CLASSICAL DEFINITION OF PROBABILITY

If there are n-mutually exclusive, exhaustive and equally likely outcomes to a random experiment and 'm' of them are favourable to an event A, then the probability of happening of A is denoted by P (A) and is defined by $P(A) = \frac{m}{n}$.

$$P(A) = \frac{\text{No. of elementary events favourable to A}}{\text{Total no. of equally likely elementary events}}$$

Obviously, $0 \le m \le n$, therefore $0 \le \frac{m}{n} \le 1$ so that $0 \le P(A) \le 1$.

P(A) can never be negative.

Since, the number of cases in which the event A will not happen is 'n – m', then the probability $P(\overline{A})$ of not happening of A is given by

$$P(\overline{A}) = \frac{n-m}{n} = 1-\frac{m}{n} = 1-P(A)$$

$$\Rightarrow \boxed{P(A) + P(\overline{A}) = 1}$$

The **ODDS IN FAVOUR** of occurrence of A are given by

m : (n – n) or $P(A) : P(\overline{A})$

The **ODDS AGAINST** the occurrence of A are given by

(n – m) : m or $P(\overline{A}) : P(A)$.

ALGEBRA OF EVENTS

Let A and B be two events related to a random experiment. We define

(i) The event "A or B" denoted by "$A \cup B$", which occurs when A or B or both occur. Thus,

$P(A \cup B)$ = Probability that at least one of the events occur

(ii) The event "A and B", denoted by "$A \cap B$", which occurs when A and B both occur. Thus,

$P(A \cap B)$ = Probability of simultaneous occurrence of A and B.

(iii) The event "Not – A" denoted by $\overline{A}$, which occurs when and only when A does not occur. Thus

$P(\overline{A})$ = Probability of non-occurrence of the event A.

(iv) $\overline{A} \cap \overline{B}$ denotes the "non-occurrence of both A and B".

(v) "$A \subset B$" denotes the "occurrence of A implies the occurrence of B".

For example:

Consider a single throw of die and following two events

A = the number is even = {2, 4, 6}

B = the number is a multiple of 3 = {3, 6}

Then $P(A \cup B) = \frac{4}{6} = \frac{2}{3}$, $P(A \cap B) = \frac{1}{6}$

$P(\overline{A}) = \frac{1}{2}$, $P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = 1 - \frac{2}{3} = \frac{1}{3}$.

ADDITION THEOREM ON PROBABILITY

1. ADDITION THEOREM: If A and B are two events associated with a random experiment, then

   $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

2. ADDITION THEOREM FOR THREE EVENTS: If A, B, C are three events associated with a random experiment, then

   $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

3. If A and B are two mutually exclusive events and the probability of their occurrence are P(A) and P(B) respectively, then probability of either A or B occurring is given by

   P(A or B) = P(A) + P(B)

   $\Rightarrow P(A + B) = P(A) + P(B)$

CONDITIONAL PROBABILITY

Let A and B be two events associated with a random experiment. Then

$P\left(\frac{A}{B}\right)$, represents the conditional probability of occurrence of A relative to B.

Also, $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$ and $P\frac{(B)}{(A)} = \frac{P(A \cap B)}{P(A)}$

For example:

Suppose a bag contains 5 white and 4 red balls. Two balls are drawn one after the other without replacement. If A denotes the event "drawing a white ball in the first draw" and B denotes the event "drawing a red ball in the second draw".

P(B/A) = Probability of drawing a red ball in second draw when it is known that a white ball has already been drawn in the first draw = $\frac{4}{8} = \frac{1}{2}$

Obviously, P(A/B) is meaning less in this problem.

MULTIPLICATION THEOREM

If A and B are two events, then

$$P(A \cap B) = P(A)\, P(B/A), \text{ if } P(A)>0$$

$$= P(B)\, P(A/B) \text{ if } P(B)>0$$

From this theorem we get

$$P(B/A) = \frac{P(A \cap B)}{P(A)} \text{ and } P(A/B) = \frac{P(A \cap B)}{P(B)}$$

For example:

Consider an experiment of throwing a pair of dice. Let A denotes the event "the sum of the point is 8" and B event "there is an even number on first die"

Then A = {(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)}

B = {2, 1), (2, 2), ………(2, 6), (4, 1), (4, 2), ……., (4, 6), (6, 1), (6, 2), ……… (6, 6)}

$$P(A) = \frac{5}{36}, P(B) = \frac{18}{36} = \frac{1}{2}, P(A \cap B) = \frac{3}{36} = \frac{1}{12}$$

Now, P(A/B) = Prob. of occurrence of A when B has already occurred = prob. of getting 8 as the sum, when there is an even number on the first die

$$= \frac{3}{18} = \frac{1}{6} \text{ and similarly } P(B/A) = \frac{3}{5}.$$

INDEPENDENCE

An event B is said to be independent of an event A if the probability that B occurs is not influenced by whether A has or has not occurred. For two independent events A and B.

$P(A \cap B) = P(A)\, P(B)$

Event $A_1, A_2, \ldots\ldots A_n$ are independent if

(i) $P(A_i \cap A_j) = P(A_i)P(A_j)$ for all i, j, $i \neq j$, That is, the events are pairwise independent.

(ii) The probability of simultaneous occurrence of (any) finite number of them is equal to the product of their separate probabilities, that is, they are mutually independent.

For example:

Let a pair of fair coin be tossed, here S = {HH, HT, TH, TT}

A = heads on the first coin = {HH, HT

B = heads on the second coin = {TH, HH}

C = heads on exactly one coin = {HT, TH}

Then $P(A) = P(B) = P(C) = \frac{2}{4} = \frac{1}{2}$ and

$P(A \cap B) = P(\{HH\}) = \frac{1}{4} = P(A)\,P(B)$

$P(B \cap C) = P(\{TH\}) = \frac{1}{4} = P(B)\,P(C)$

$P(A \cap C) = P(\{HT\}) = \frac{1}{4} = P(A)\,P(C)$

Hence the events are pairwise independent.

Also $P(A \cap B \cap C) = P(\phi) = 0 \neq P(A)\,P(B)\,P(C)$

Hence, the events A, B, C are not mutually independent.